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Mathematics HL and further mathematics HL formula booklet

For use during the course and in the examinations
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Formulae

Prior learning

Area of a parallelogram	$A = b \times h$, where b is the base, h is the height
Area of a triangle	$A = \frac{1}{2}(b \times h)$, where b is the base, h is the height
Area of a trapezium	$A = \frac{1}{2}(a + b)h$, where a and b are the parallel sides, h is the height
Area of a circle	$A = \pi r^2$, where r is the radius
Circumference of a circle	$C = 2\pi r$, where r is the radius
Volume of a pyramid	$V = \frac{1}{3}(\text{area of base} \times \text{vertical height})$
Volume of a cuboid	$V = l \times w \times h$, where l is the length, w is the width, h is the height
Volume of a cylinder	$V = \pi r^2 h$, where r is the radius, h is the height
Area of the curved surface of a cylinder	$A = 2\pi r h$, where r is the radius, h is the height
Volume of a sphere	$V = \frac{4}{3}\pi r^3$, where r is the radius
Volume of a cone	$V = \frac{1}{3}\pi r^2 h$, where r is the radius, h is the height
Distance between two points (x_1, y_1) and (x_2, y_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Coordinates of the midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$
Solutions of a quadratic equation	The solutions of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Topic I: Algebra

1.1	<p>The nth term of an arithmetic sequence</p> <p>The sum of n terms of an arithmetic sequence</p> <p>The nth term of a geometric sequence</p> <p>The sum of n terms of a finite geometric sequence</p> <p>The sum of an infinite geometric sequence</p>	$u_n = u_1 + (n-1)d$ $S_n = \frac{n}{2}(2u_1 + (n-1)d) = \frac{n}{2}(u_1 + u_n)$ $u_n = u_1 r^{n-1}$ $S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}, r \neq 1$ $S_\infty = \frac{u_1}{1 - r}, r < 1$
1.2	Exponents and logarithms	$a^x = b \Leftrightarrow x = \log_a b, \text{ where } a > 0, b > 0, a \neq 1$ $a^x = e^{x \ln a}$ $\log_a a^x = x = a^{\log_a x}$ $\log_b a = \frac{\log_c a}{\log_c b}$
1.3	<p>Combinations</p> <p>Permutations</p> <p>Binomial theorem</p>	$\binom{n}{r} = \frac{n!}{r!(n-r)!}$ ${}^n P_r = \frac{n!}{(n-r)!}$ $(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \dots + \binom{n}{r} a^{n-r} b^r + \dots + b^n$
1.5	Complex numbers	$z = a + ib = r(\cos \theta + i \sin \theta) = r e^{i\theta} = r \operatorname{cis} \theta$
1.7	De Moivre's theorem	$[r(\cos \theta + i \sin \theta)]^n = r^n (\cos n\theta + i \sin n\theta) = r^n e^{in\theta} = r^n \operatorname{cis} n\theta$

Topic 2: Functions and equations

2.5	Axis of symmetry of the graph of a quadratic function	$f(x) = ax^2 + bx + c \Rightarrow$ axis of symmetry $x = -\frac{b}{2a}$
2.6	Discriminant	$\Delta = b^2 - 4ac$

Topic 3: Circular functions and trigonometry

3.1	Length of an arc Area of a sector	$l = \theta r$, where θ is the angle measured in radians, r is the radius $A = \frac{1}{2}\theta r^2$, where θ is the angle measured in radians, r is the radius
3.2	Identities Pythagorean identities	$\tan \theta = \frac{\sin \theta}{\cos \theta}$ $\sec \theta = \frac{1}{\cos \theta}$ $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$ $\cos^2 \theta + \sin^2 \theta = 1$ $1 + \tan^2 \theta = \sec^2 \theta$ $1 + \cot^2 \theta = \operatorname{csc}^2 \theta$
3.3	Compound angle identities Double angle identities	$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$ $\sin 2\theta = 2 \sin \theta \cos \theta$ $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$ $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$

3.7	Cosine rule	$c^2 = a^2 + b^2 - 2ab \cos C$; $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$
	Sine rule	$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
	Area of a triangle	$A = \frac{1}{2} ab \sin C$

Topic 4: Vectors

4.1	Magnitude of a vector	$ \mathbf{v} = \sqrt{v_1^2 + v_2^2 + v_3^2}$, where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
	Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
	Coordinates of the midpoint of a line segment with endpoints (x_1, y_1, z_1) , (x_2, y_2, z_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$
4.2	Scalar product	$\mathbf{v} \cdot \mathbf{w} = \mathbf{v} \mathbf{w} \cos \theta$, where θ is the angle between \mathbf{v} and \mathbf{w}
	Angle between two vectors	$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$, where $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $\cos \theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{ \mathbf{v} \mathbf{w} }$
4.3	Vector equation of a line	$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$
	Parametric form of the equation of a line	$x = x_0 + \lambda l$, $y = y_0 + \lambda m$, $z = z_0 + \lambda n$
	Cartesian equations of a line	$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$

4.5	Vector product	$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix} \text{ where } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$
	Area of a triangle	$ \mathbf{v} \times \mathbf{w} = \mathbf{v} \mathbf{w} \sin \theta, \text{ where } \theta \text{ is the angle between } \mathbf{v} \text{ and } \mathbf{w}$ $A = \frac{1}{2} \mathbf{v} \times \mathbf{w} \text{ where } \mathbf{v} \text{ and } \mathbf{w} \text{ form two sides of a triangle}$
4.6	Vector equation of a plane	$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b} + \mu \mathbf{c}$
	Equation of a plane (using the normal vector)	$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$
	Cartesian equation of a plane	$ax + by + cz = d$

Topic 5: Statistics and probability

5.1	Population parameters	Let $n = \sum_{i=1}^k f_i$
	Mean μ	$\mu = \frac{\sum_{i=1}^k f_i x_i}{n}$
	Variance σ^2	$\sigma^2 = \frac{\sum_{i=1}^k f_i (x_i - \mu)^2}{n} = \frac{\sum_{i=1}^k f_i x_i^2}{n} - \mu^2$
	Standard deviation σ	$\sigma = \sqrt{\frac{\sum_{i=1}^k f_i (x_i - \mu)^2}{n}}$
5.2	Probability of an event A	$P(A) = \frac{n(A)}{n(U)}$
	Complementary events	$P(A) + P(A') = 1$
5.3	Combined events	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
	Mutually exclusive events	$P(A \cup B) = P(A) + P(B)$

5.4	Conditional probability	$P(A B) = \frac{P(A \cap B)}{P(B)}$
	Independent events	$P(A \cap B) = P(A) P(B)$
	Bayes' theorem	$P(B A) = \frac{P(B) P(A B)}{P(B) P(A B) + P(B') P(A B')}$
		$P(B_i A) = \frac{P(B_i) P(A B_i)}{P(B_1) P(A B_1) + P(B_2) P(A B_2) + P(B_3) P(A B_3)}$
5.5	Expected value of a discrete random variable X	$E(X) = \mu = \sum x P(X = x)$
	Expected value of a continuous random variable X	$E(X) = \mu = \int_{-\infty}^{\infty} x f(x) dx$
	Variance	$\text{Var}(X) = E(X - \mu)^2 = E(X^2) - [E(X)]^2$
	Variance of a discrete random variable X	$\text{Var}(X) = \sum (x - \mu)^2 P(X = x) = \sum x^2 P(X = x) - \mu^2$
	Variance of a continuous random variable X	$\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx = \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2$
5.6	Binomial distribution	$X \sim B(n, p) \Rightarrow P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \dots, n$
	Mean	$E(X) = np$
	Variance	$\text{Var}(X) = np(1-p)$
	Poisson distribution	$X \sim \text{Po}(m) \Rightarrow P(X = x) = \frac{m^x e^{-m}}{x!}, \quad x = 0, 1, 2, \dots$
	Mean	$E(X) = m$
	Variance	$\text{Var}(X) = m$
5.7	Standardized normal variable	$z = \frac{x - \mu}{\sigma}$

Topic 6: Calculus

6.1	Derivative of $f(x)$	$y = f(x) \Rightarrow \frac{dy}{dx} = f'(x) = \lim_{h \rightarrow 0} \left(\frac{f(x+h) - f(x)}{h} \right)$
6.2	Derivative of x^n	$f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$
	Derivative of $\sin x$	$f(x) = \sin x \Rightarrow f'(x) = \cos x$
	Derivative of $\cos x$	$f(x) = \cos x \Rightarrow f'(x) = -\sin x$
	Derivative of $\tan x$	$f(x) = \tan x \Rightarrow f'(x) = \sec^2 x$
	Derivative of e^x	$f(x) = e^x \Rightarrow f'(x) = e^x$
	Derivative of $\ln x$	$f(x) = \ln x \Rightarrow f'(x) = \frac{1}{x}$
	Derivative of $\sec x$	$f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x$
	Derivative of $\csc x$	$f(x) = \csc x \Rightarrow f'(x) = -\csc x \cot x$
	Derivative of $\cot x$	$f(x) = \cot x \Rightarrow f'(x) = -\csc^2 x$
	Derivative of a^x	$f(x) = a^x \Rightarrow f'(x) = a^x (\ln a)$
	Derivative of $\log_a x$	$f(x) = \log_a x \Rightarrow f'(x) = \frac{1}{x \ln a}$
	Derivative of $\arcsin x$	$f(x) = \arcsin x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}$
	Derivative of $\arccos x$	$f(x) = \arccos x \Rightarrow f'(x) = -\frac{1}{\sqrt{1-x^2}}$
	Derivative of $\arctan x$	$f(x) = \arctan x \Rightarrow f'(x) = \frac{1}{1+x^2}$
	Chain rule	$y = g(u), \text{ where } u = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
	Product rule	$y = uv \Rightarrow \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$
	Quotient rule	$y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

6.4	Standard integrals	$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$ $\int \frac{1}{x} dx = \ln x + C$ $\int \sin x dx = -\cos x + C$ $\int \cos x dx = \sin x + C$ $\int e^x dx = e^x + C$ $\int a^x dx = \frac{1}{\ln a} a^x + C$ $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$ $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin\left(\frac{x}{a}\right) + C, \quad x < a$
6.5	<p>Area under a curve</p> <p>Volume of revolution (rotation)</p>	$A = \int_a^b y dx \text{ or } A = \int_a^b x dy$ $V = \int_a^b \pi y^2 dx \text{ or } V = \int_a^b \pi x^2 dy$
6.7	Integration by parts	$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx \text{ or } \int u dv = uv - \int v du$

Topic 7: Statistics and probability

Further mathematics HL topic 3

7.1 (3.1)	Probability generating function for a discrete random variable X	$G(t) = E(t^x) = \sum_x P(X = x)t^x$ $E(X) = G'(1)$ $\text{Var}(X) = G''(1) + G'(1) - (G'(1))^2$
7.2 (3.2)	Linear combinations of two independent random variables X_1, X_2	$E(a_1X_1 \pm a_2X_2) = a_1E(X_1) \pm a_2E(X_2)$ $\text{Var}(a_1X_1 \pm a_2X_2) = a_1^2 \text{Var}(X_1) + a_2^2 \text{Var}(X_2)$
7.3 (3.3)	Sample statistics	
	Mean \bar{x}	$\bar{x} = \frac{\sum_{i=1}^k f_i x_i}{n}$
	Variance s_n^2	$s_n^2 = \frac{\sum_{i=1}^k f_i (x_i - \bar{x})^2}{n} = \frac{\sum_{i=1}^k f_i x_i^2}{n} - \bar{x}^2$
	Standard deviation s_n	$s_n = \sqrt{\frac{\sum_{i=1}^k f_i (x_i - \bar{x})^2}{n}}$
	Unbiased estimate of population variance s_{n-1}^2	$s_{n-1}^2 = \frac{n}{n-1} s_n^2 = \frac{\sum_{i=1}^k f_i (x_i - \bar{x})^2}{n-1} = \frac{\sum_{i=1}^k f_i x_i^2}{n-1} - \frac{n}{n-1} \bar{x}^2$
7.5 (3.5)	Confidence intervals	
	Mean, with known variance	$\bar{x} \pm z \times \frac{\sigma}{\sqrt{n}}$
	Mean, with unknown variance	$\bar{x} \pm t \times \frac{s_{n-1}}{\sqrt{n}}$
7.6 (3.6)	Test statistics	
	Mean, with known variance	$z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}}$

	Mean, with unknown variance	$t = \frac{\bar{x} - \mu}{s_{n-1} / \sqrt{n}}$
7.7 (3.7)	Sample product moment correlation coefficient	$r = \frac{\sum_{i=1}^n x_i y_i - n\bar{x}\bar{y}}{\sqrt{\left(\sum_{i=1}^n x_i^2 - n\bar{x}^2\right)\left(\sum_{i=1}^n y_i^2 - n\bar{y}^2\right)}}$
	Test statistic for $H_0: \rho = 0$	$t = r\sqrt{\frac{n-2}{1-r^2}}$
	Equation of regression line of x on y	$x - \bar{x} = \left(\frac{\sum_{i=1}^n x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^n y_i^2 - n\bar{y}^2} \right) (y - \bar{y})$
	Equation of regression line of y on x	$y - \bar{y} = \left(\frac{\sum_{i=1}^n x_i y_i - n\bar{x}\bar{y}}{\sum_{i=1}^n x_i^2 - n\bar{x}^2} \right) (x - \bar{x})$

Topic 8: Sets, relations and groups

Further mathematics HL topic 4

8.1 (4.1)	De Morgan's laws	$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$
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Topic 9: Calculus

Further mathematics HL topic 5

9.5 (5.5)	Euler's method	$y_{n+1} = y_n + h \times f(x_n, y_n); \quad x_{n+1} = x_n + h, \text{ where } h \text{ is a constant (step length)}$
	Integrating factor for $y' + P(x)y = Q(x)$	$e^{\int P(x) dx}$

9.6 (5.6)	Maclaurin series	$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots$
	Taylor series	$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$
	Taylor approximations (with error term $R_n(x)$)	$f(x) = f(a) + (x-a) f'(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + R_n(x)$
	Lagrange form	$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}, \text{ where } c \text{ lies between } a \text{ and } x$
	Maclaurin series for special functions	$e^x = 1 + x + \frac{x^2}{2!} + \dots$ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

Topic 10: Discrete mathematics

Further mathematics HL topic 6

10.7 (6.7)	Euler's formula for connected planar graphs	$v - e + f = 2$, where v is the number of vertices, e is the number of edges, f is the number of faces
	Planar, simple, connected graphs	$e \leq 3v - 6 \text{ for } v \geq 3$ $e \leq 2v - 4 \text{ if the graph has no triangles}$

Formulae for distributions

Topics 5.6, 5.7, 7.1, further mathematics HL topic 3.1

Discrete distributions

Distribution	Notation	Probability mass function	Mean	Variance
Geometric	$X \sim \text{Geo}(p)$	pq^{x-1} for $x = 1, 2, \dots$	$\frac{1}{p}$	$\frac{q}{p^2}$
Negative binomial	$X \sim \text{NB}(r, p)$	$\binom{x-1}{r-1} p^r q^{x-r}$ for $x = r, r+1, \dots$	$\frac{r}{p}$	$\frac{rq}{p^2}$

Continuous distributions

Distribution	Notation	Probability density function	Mean	Variance
Normal	$X \sim N(\mu, \sigma^2)$	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$	μ	σ^2

1.2	Determinant of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \det A = A = ad - bc$
	Inverse of a 2×2 matrix	$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, ad \neq bc$
	Determinant of a 3×3 matrix	$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \Rightarrow \det A = a \begin{vmatrix} e & f \\ h & k \end{vmatrix} - b \begin{vmatrix} d & f \\ g & k \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$